Mathematical Modeling and Analysis

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A family of mimetic finite difference methods on polygonal and polyhedral meshes

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In many applications, the mathematical model is formulated initially as a system of first-order partial differential equations, with each equation having a natural connection to physical aspects of the problem. Let us consider the linear diffusion problem:

$$\operatorname{div} \vec{F} = b, \quad \vec{F} = -\mathbb{K} \operatorname{grad} p$$

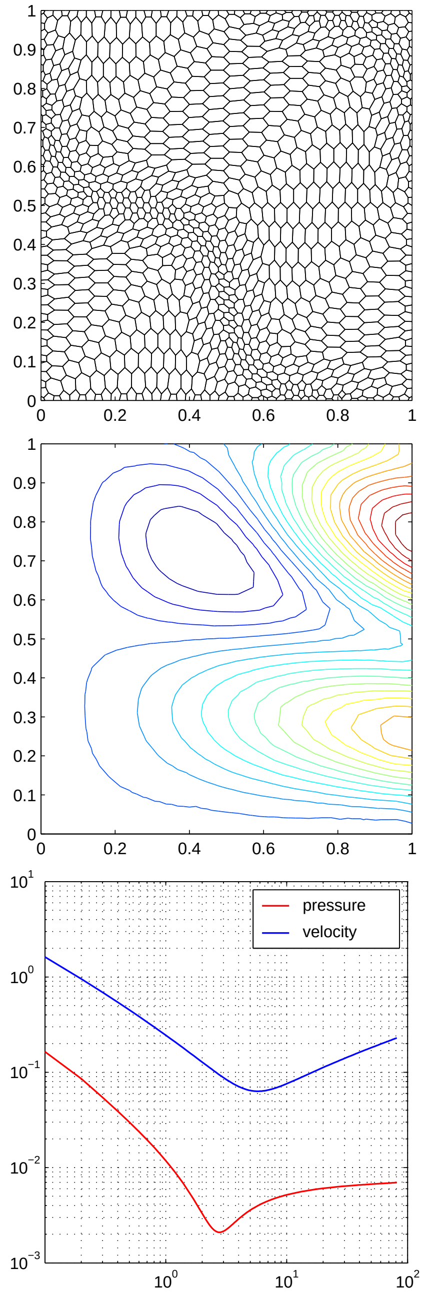
where the first equation describes the mass conservation and the second one is the constitutive equation relating the scalar function p to the velocity field $\vec{F}$. The material properties are described by $\mathbb{K}$ which is generally a full symmetric tensor.

The mimetic finite difference (MFD) method [1] has been successfully employed for solving this problem on simplicial, quadrilateral, hexahedral, and unstructured polygonal and polyhedral meshes in both Cartesian and cylindrical coordinate systems.

It was shown in [2] that the convergence analysis of the MFD method is reduced to an algebraic equation with an unknown matrix. More precisely, for each mesh element E with s_E faces (edges in 2D), we are looking for a symmetric positive definite $s_E \times s_E$ matrix $\mathbb{W}_E$ satisfying

$$\mathbb{W}_E \, \mathbb{R}_E = \mathbb{N}_E \, \mathbb{K}_E \,.$$

Here $\mathbb{K}_E$ is the tensor value at the center of mass of E and matrices $\mathbb{N}_E$ and $\mathbb{R}_E$ are uniquely defined by geometry of E. The s-th row of matrix $\mathbb{N}_E$ is the unit normal to the s-th face of E. The s-th row of matrix $\mathbb{R}_E$ is the center of mass of the s-th face multiplied by its area (length in 2D).



A polygonal mesh, solution isolines and the L_2-norms of errors in primary variables p and $\vec{F}$ as functions of u/u_E.

In [3], we employed an innovative technique to give a rigorous mathematic description of a family of acceptable solutions of the matrix equation.

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Each solution results in a MFD method with the optimal convergence rate on unstructured polygonal and polyhedral meshes with almost flat faces.

We developed a very simple algorithm for computing a few particular solutions of the matrix equation. They are given by

$$\mathbb{W}_E = \frac{1}{|E|}\mathbb{N}_E\,\mathbb{K}_E\,\mathbb{N}_E^T + u\,\mathbb{D}_E$$

where $\mathbb{D}_E$ is an easily computable orthogonal projector, $|E|$ is the volume (area in 2D) of E and u is a positive number. In practice, we recommend to set u close to $u_E = \text{trace}(\mathbb{K}_E)/|E|$ to balance the terms.

The complexity of this algorithm is proportional to s_E^2, the number of entries in the matrix $\mathbb{W}_E$. With the new method, discretizations on polygonal and polyhedral meshes look as simple as discretizations on triangular and tetrahedral meshes.

In the illustrative example, we solve the Dirichlet boundary value problem with a full tensor

$$\mathbb{K} = \begin{pmatrix} (x+1)^2 + y^2 & -xy \\ -xy & (x+1)^2 \end{pmatrix}.$$

The isolines of the exact solution are shown on the middle picture. The bottom picture shows errors (vertical axis) as functions of u/u_E (horizontal axis) for the presented polygonal mesh. Note, that there is a big interval $u/u_E \in [2, 80]$ where the errors vary only 3 times. What is remarkable here is that for all values of u we observed second order convergence rate for the scalar variable and 1.5 convergence rate for the vector variable.

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